

A norm inequality for pairs of commuting positive semidefinite matrices

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Abstract

For $k = 1, \dots, K$, let A_k and B_k be positive semidefinite matrices such that, for each k , A_k commutes with B_k . We show that, for any unitarily invariant norm,

$$||| \sum_{k=1}^K A_k B_k ||| \leq ||| \left(\sum_{k=1}^K A_k \right) \left(\sum_{k=1}^K B_k \right) |||.$$

In this paper, we denote the vectors of eigenvalues and singular values of a matrix A by $\lambda(A)$ and $\sigma(A)$, respectively. We adhere to the convention to sort singular values, and eigenvalues as well whenever they are real, in non-increasing order. In general, for a real vector x , we will write x^\downarrow for the vector with the same components as x but sorted in non-increasing order.

For real n -dimensional vectors x and y , we say that x is *weakly majorised* by y , denoted $x \prec_w y$, if and only if for $k = 1, \dots, n$, $\sum_{i=1}^k x_i^\downarrow \leq \sum_{i=1}^k y_i^\downarrow$. We say that x is *majorised* by y , denoted $x \prec y$, if and only if $x \prec_w y$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. If, moreover, x and y are non-negative, we say that x is *weakly log-majorised* by y , denoted $x \prec_{w,\log} y$, if and only if for $k = 1, \dots, n$, $\prod_{i=1}^k x_i^\downarrow \leq \prod_{i=1}^k y_i^\downarrow$.

According to Weyl's Majorant Theorem ([1] Theorem II.3.6, or [3], Theorem 2.4), the vector of singular values of any matrix log-majorises the vector of the absolute values of its eigenvalues: $|\lambda(A)| \prec_{\log} \sigma(A)$. As $x \prec_{w,\log} y$ implies

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$x^r \prec_w y^r$ for any $r > 0$, Weyl's Majorant Theorem can in slightly weaker form be stated as

$$|\lambda(A)|^r \prec_w \sigma^r(A). \quad (1)$$

The sum of the k largest singular values of a matrix defines a norm, known as the k -th Ky Fan norm. The convexity of the Ky Fan norms can be expressed as a majorisation relation: for any p such that $0 \leq p \leq 1$,

$$\sigma(pA + (1-p)B) \prec_w p\sigma(A) + (1-p)\sigma(B).$$

When A and B are positive semidefinite, their singular values coincide with their eigenvalues and we have

$$\lambda(pA + (1-p)B) \prec p\lambda(A) + (1-p)\lambda(B). \quad (2)$$

For positive semidefinite matrices A and B , the eigenvalues of AB are real and non-negative. Furthermore $\lambda(AB) \prec_{\log} \lambda(A)\lambda(B)$ ([3] eq. (2.5)). Hence, we also have

$$\lambda(AB) \prec_w \lambda(A)\lambda(B). \quad (3)$$

We start with a rather technical result concerning a majorisation relation for singular values:

Lemma 1 *Let S be a general $n \times m$ complex matrix, and L and M two diagonal, positive semidefinite $m \times m$ matrices. Then*

$$\sigma(SL \text{Diag}(S^*S)MS^*) \prec_w \sigma(SLS^*SM S^*). \quad (4)$$

Proof. We first show that

$$\sigma(SL \text{Diag}(S^*S)MS^*) \prec_w \sigma(S(LM)^{1/2}S^*S(LM)^{1/2}S^*). \quad (5)$$

Since L , M , and $\text{Diag}(S^*S)$ are diagonal, they commute, and we can write $SL \text{Diag}(S^*S)MS^* = S(LM)^{1/2} \text{Diag}(S^*S)(LM)^{1/2}S^*$. This is a positive semidefinite matrix, hence its singular values are equal to its eigenvalues. The same is true for the right-hand side of (5). Let us introduce $X = S(LM)^{1/4}$ and $T = X^*X \geq 0$. Then we have to show $\lambda(X \text{Diag}(X^*X)X^*) \prec \lambda(XX^*XX^*)$, or $\lambda(T \text{Diag}(T)) \prec \lambda(T^2)$. Now note that there exist J unitary matrices U_j such that $\text{Diag}(T) = \sum_{j=1}^J (U_j T U_j^*)/J$. Exploiting (2) and inequality (3) in turn, we obtain

$$\begin{aligned}
\lambda(T \operatorname{Diag}(T)) &= \lambda(T^{1/2} \sum_{j=1}^J \frac{1}{J} (U_j T U_j^*) T^{1/2}) \\
&\prec \sum_{j=1}^J \frac{1}{J} \lambda(T^{1/2} U_j T U_j^* T^{1/2}) \\
&= \sum_{j=1}^J \frac{1}{J} \lambda(T U_j T U_j^*) \\
&\prec_w \sum_{j=1}^J \frac{1}{J} \lambda(T) \lambda(U_j T U_j^*) \\
&= \sum_{j=1}^J \frac{1}{J} \lambda^2(T) = \lambda(T^2),
\end{aligned}$$

which proves (5).

Secondly, we show that

$$\sigma(S(LM)^{1/2} S^* S(LM)^{1/2} S^*) \prec_w \sigma(SLS^* SMS^*). \quad (6)$$

Since $(LM)^{1/2}$ and S^*S are both positive semidefinite, their matrix product has real, non-negative eigenvalues. Thus,

$$\begin{aligned}
\lambda^2((LM)^{1/2} S^* S) &= |\lambda(L^{1/2} S^* S M^{1/2})|^2 \\
&\prec_w \sigma^2(L^{1/2} S^* S M^{1/2}),
\end{aligned}$$

by Weyl's Majorant Theorem (eq. (1) with $r = 2$). This implies (6):

$$\begin{aligned}
\sigma(S(LM)^{1/2} S^* S(LM)^{1/2} S^*) &= \lambda((LM)^{1/2} S^* S(LM)^{1/2} S^* S) \\
&= \lambda^2((LM)^{1/2} S^* S) \\
&\prec_w \sigma^2(L^{1/2} S^* S M^{1/2}) \\
&= \lambda^2((M^{1/2} S^* S L S^* S M^{1/2})^{1/2}) \\
&= \lambda(M^{1/2} S^* S L S^* S M^{1/2}) \\
&= \lambda(SLS^* SMS^*) \\
&= |\lambda(SLS^* SMS^*)| \\
&\prec_w \sigma(SLS^* SMS^*),
\end{aligned}$$

where in the last line we again exploited Weyl's Majorant Theorem (eq. (1) with $r = 1$).

Combining (5) with (6) yields (4). \square

We can now state and prove the main result of this paper.

Theorem 1 *For $k = 1, \dots, K$, let A_k and B_k be positive semidefinite $d \times d$ matrices such that, for each k , A_k commutes with B_k . Then for all unitarily invariant norms*

$$||| \sum_{k=1}^K A_k B_k ||| \leq ||| (\sum_{k=1}^K A_k) (\sum_{k=1}^K B_k) |||. \quad (7)$$

Proof. Let A_k and B_k have eigenvalue decompositions

$$A_k = U_k a_k U_k^*, \quad B_k = U_k b_k U_k^*,$$

where the U_k are unitary matrices, and a_k and b_k are positive semidefinite diagonal matrices. Let

$$L = \bigoplus_{k=1}^K a_k, \quad M = \bigoplus_{k=1}^K b_k, \quad S = (U_1 | U_2 | \dots | U_K).$$

Then

$$\sum_{k=1}^K A_k = S L S^*, \quad \sum_{k=1}^K B_k = S M S^*, \quad \sum_{k=1}^K A_k B_k = S L M S^*.$$

In addition, $\text{Diag}(S^* S) = \mathbb{I}$ since all columns of S are normalised. By Lemma 1, we then have

$$\sigma\left(\sum_{k=1}^K A_k B_k\right) \prec_w \sigma\left(\left(\sum_{k=1}^K A_k\right) \left(\sum_{k=1}^K B_k\right)\right)$$

which is equivalent to (7). \square

A simple consequence of Theorem 1 is that for any set of K positive semidefinite matrices A_k , all positive functions f and g , and all unitarily invariant norms,

$$||| \sum_{k=1}^K f(A_k) g(A_k) ||| \leq ||| \left(\sum_{k=1}^K f(A_k)\right) \left(\sum_{k=1}^K g(A_k)\right) |||. \quad (8)$$

Setting $K = 2$, $f(x) = x^p$ and $g(x) = x^q$ yields the inequality

$$||| A^{p+q} + B^{p+q} ||| \leq ||| (A^p + B^p)(A^q + B^q) |||, \quad (9)$$

which was recently conjectured by Bourin [2].

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